

Solutions for the Branched Case

4-1. Up to this point we have looked at solutions for the cylinder and the sphere. However, neurons have extensive branching. How do we incorporate branching into the solutions?

One way to include branching is to consider additional cylinders as the “leak conductance” added on to the end of our initial cylinder. Consider the solution on a cable with current input at one end and a leaky end at the other end. What are the boundary conditions? The boundary conditions are $-1/r_i dV/dx(x=0) = I_0$ and $-1/r_i dV/dx(x=\ell) = VG_L$. The solution is:

$$V(x) = \frac{I_0 \lambda r_i \left[\cosh\left(\frac{\ell-x}{\lambda}\right) + \frac{G_L}{G_\infty} \sinh\left(\frac{\ell-x}{\lambda}\right) \right]}{\sinh\left(\frac{\ell}{\lambda}\right) + \frac{G_L}{G_\infty} \cosh\left(\frac{\ell}{\lambda}\right)} \quad \text{where } G_\infty = 1/(\lambda r_i) \text{ represents the conductance}$$

of an infinite extension of the cylinder with same diameter.

If $G_L=0$ (sealed end condition) we get the solution on overhead 3-9.

If $G_L=G_\infty$ then the leak is equivalent to an infinite extension of the cylinder and

$$V(x) = I_0 / G_\infty \exp(-x/\lambda)$$

If $G_L=\infty$ then the end is voltage clamped to 0. It is left as an exercise to verify these latter two claims. You can substitute in the solutions to verify this.

4-2. Now what might $G_L / G_\infty = 2$ correspond to anatomically? This might correspond to two or more branches connected at the end of the cable, a cable that flares, or a single cable with a step increase in diameter.

How about $G_L / G_\infty = 0.5$? What does this correspond to anatomically? This might correspond to one or more small branches attached, one branch with taper attached, or one branch with a step decrease in diameter. So it seems possible to incorporate multiple branches into a solution by choosing the end boundary condition appropriately.

Recall our solution

$$V(x) = \frac{I_0 \lambda r_i \left[\cosh\left(\frac{\ell-x}{\lambda}\right) + \frac{G_L}{G_\infty} \sinh\left(\frac{\ell-x}{\lambda}\right) \right]}{\sinh\left(\frac{\ell}{\lambda}\right) + \frac{G_L}{G_\infty} \cosh\left(\frac{\ell}{\lambda}\right)}. \quad \text{Suppose } G_L \text{ (the leak conductance at the end of}$$

the parent cylinder) equals the input conductance of an attached cylinder, G_{in}' , and this attached cylinder has a leak conductance at its end, G_L' , where G_L' could represent more attached cylinders. Then $G_L = G_{in}' = I_0/V_2(0)$ which can be computed from the above equation. If we do the calculation and divide the numerator and denominator by $\cosh(\ell/\lambda)$, then we get

$G_L = G'_{in} = \frac{I_0}{V_2(0)} = \frac{G'_\infty (G'_L/G'_\infty + \tanh(L))}{1 + G'_L/G'_\infty \tanh(L)}$ where $L = \frac{\ell}{\lambda}$. Now G'_L = the leak condition at the end of the attached cylinder and G'_∞ equals “ G_∞ ” for this attached cylinder (or the input conductance of an infinite extension of cylinder 2).

4-3. Rall developed an iterative algorithm for computing the voltage in branched structures, and in this algorithm he assigns $B = G'_L / G'_\infty$. Then we have

$$G_L = G'_{in} = \frac{\pi}{2} \frac{d^{3/2}}{\sqrt{R_m R_a}} \left[\frac{B + \tanh(\ell/\lambda)}{1 + B \tanh(\ell/\lambda)} \right]$$

where the term before the brackets is G_∞ and the term in the brackets is the adjustment in the conductance from G_∞ because of branching and the fact that the cable is finite in length.

Now the input conductance of the attached cylinder depends on the terminal boundary condition. This terminal boundary condition could be more cylinders. In Rall’s algorithm you start with the terminal branch and

- 1) Give B, calculate $G_L = G'_{in}$ (for the end cylinder)
- 2) Then given G_L , calculate G_{in} for the next to last or parent cylinder.
- 3) This G_{in} becomes the G_L for the next to next to last cylinder, etc.

The general procedure is to calculate the input conductance iteratively from the terminal boundaries in toward the soma. Then for a given I_0 , the voltage can be computed everywhere.

Rall (1959). Assume the terminal boundary condition is sealed end. This means $B=0$.

With $B=0$, $G'_{in} = G'_\infty \tanh(L)$

$G'_{in} = G_L$ for the next cylinder

So the next $B = G_L / G_\infty$

4-4. What if there are multiple cylinders? Then the G_L = the sum of the branch G_{in} values.

Consider the example in this overhead where branches 1 and 2 are connected on the left to branch 3 and branches 3 and 4 are connected on the left to branch 5. The terminal branches (branches that end on the right) are branches 1, 2, and 4. If we assume sealed end at these terminals, then $B=0$ for these cylinders.

Then the input conductances for cylinders 1 and 2 are

$$G_1 = G_{1\infty} \tanh(L_1) \text{ and } G_2 = G_{2\infty} \tanh(L_2)$$

$$\text{where } G_{1\infty} = \pi/2 d_1^{3/2} (R_m R_a)^{-1/2} \text{ and } G_{2\infty} = \pi/2 d_2^{3/2} (R_m R_a)^{-1/2}$$

So the G_L value for cylinder 3 = $G_1 + G_2$. So the input conductance for cylinder G_3 is

$$G_3 = \frac{I_3}{V(0)} = \frac{G_{3\infty} \left(\frac{(G_1 + G_2)}{G_{3\infty}} + \tanh(L_3) \right)}{1 + \frac{(G_1 + G_2)}{G_{3\infty}} \tanh(L_3)} \text{ where } G_{L3} = G_1 + G_2$$

Now cylinder 4 is a terminal branch so its input conductance is

$$G_4 = G_{4\infty} \tanh(L_4) \text{ because the end is assumed to be a sealed end.}$$

4-5. Then the G_L value for cylinder 5 = $G_3 + G_4$ and

$$G_5 = \frac{I_5}{V(0)} = \frac{G_{5\infty} \left(\frac{(G_3 + G_4)}{G_{5\infty}} + \tanh(L_5) \right)}{1 + \frac{(G_3 + G_4)}{G_{5\infty}} \tanh(L_5)}. \text{ Since we assume we know the current injected into cylinder 5}$$

at $x=0$, we can compute $V_5(0)$. $V_5(0) = I_5/G_5$

Now we consider the solution of the cable equation with voltage clamp at $x=0$ and leaky end at $x=\ell$. In the current notation, this becomes,

$$V_5(x) = V_5(0) \frac{\cosh(L_5 - X) + B \sinh(L_5 - X)}{\cosh(L_5) + B \sinh(L_5)} \text{ where } B = (G_3 + G_4)/G_{5\infty} \text{ (} G_L/G_{\infty}\text{)}.$$

From this we get $V_5(L_5) = V_4(0) = V_3(0)$.

Then we use the above solution for $V_4(x)$ and $V_3(x)$ with the appropriate B values as calculated above. Repeat and iterate to get expressions for V everywhere.

4-6. There is an alternate way to compute the steady-state solution with dendritic branching. This method involves solving n equations in n unknowns where the unknowns are the voltages at nodes (branch points and terminations). It is not essential that you know this, so if you have had enough math for the moment, skip to 4-7 below.

This solution starts with the third steady-state solution

$$V(x) = C_1 \cosh((\ell - x)/\lambda) + C_2 \sinh((\ell - x)/\lambda)$$

Let $G_0 = V(0) = C_1 \cosh(\ell/\lambda) + C_2 \sinh(\ell/\lambda)$ and $G_1 = V(\ell) = C_1$.

Very important—G is not conductance here, just voltage at a node.

Then if we solve for C_1 and C_2 in terms of G_0 and G_1 we get

$$V(x) = G_1 \cosh((\ell - x)/\lambda) + (G_0 - G_1 \cosh(\ell/\lambda)) / \sinh(\ell/\lambda) \sinh((\ell - x)/\lambda)$$

We can have a $V_i(x)$ for every cylinder i in the dendritic tree of this form.

Then we apply conservation of current at the branch points, or

$$-1/r_{i1} dV_1/dx = -1/r_{i2} dV_2/dx - 1/r_{i3} dV_3/dx \text{ all evaluated at } x=\ell_1 \text{ which is } x=0 \text{ for branches 2 and 3.}$$

and apply sealed end conditions at terminal branches and a current clamp at $x=0$ of the the first branch and this gives us a system of equations where the equations are of one of 3 types.

For a branch point we have

$$\frac{-G_{i-1}}{\sinh(\ell_i/\lambda_i)} + G_i \sum_{\substack{k=\text{parents} \\ \&\text{daughters}}} \frac{\lambda_i r_i}{\lambda_k r_k} \frac{1}{\tanh(\ell_k/\lambda_k)} - \sum_{j=\text{daughters}} G_j \frac{\lambda_i r_i}{\lambda_j r_j} \frac{1}{\sinh(\ell_j/\lambda_j)} = 0$$

for terminal nodes we have

$$\frac{-G_{i-1}}{\sinh(\ell_i/\lambda_i)} + \frac{G_i}{\tanh(\ell_i/\lambda_i)} = 0$$

and for the very first node we have

$$\frac{-G_0}{\tanh(\ell_1/\lambda_1)} + \frac{G_1}{\sinh(\ell_1/\lambda_1)} = r_1 \lambda_1 I_0$$

This forms a matrix system of the form $AG=b$ where A is a matrix of the coefficients of the G_i terms, G is a vector of the G_i terms and b is a vector of zeros except for the equation for the first node which is the right hand side of the last equation. This is solved for the G_i terms and then the $V_i(x)$ are obtained from the solution above in terms of the G_i .

4-7. Why have we looked at these solutions? What do they tell us? In this overhead is a figure from Rall reproduced in the BoG to show the voltage spread (both steady-state and transient) in a dendritic tree for input at the soma and at a terminal dendrite.

When current is injected at point I in the terminal branch, the voltage is given by the solid line in this figure. Note the extreme attenuation in the proximal direction to the next branch point and note the very little attenuation in the distal direction from the branch points. In the left figure, the injected current was a constant current and the voltage values are the steady-state voltages.

The dotted line is the voltage when the site of current injection is the soma. Note that although the current is the same as that injected in the distal terminal, the voltage change at the site of current injection is much smaller when the site is the soma. Also note that the attenuation of voltage from the soma to the terminal dendrite is very slight.

On the right we see the responses to a transient current injection at point I in the distal terminal. The attenuation is more severe for transient inputs than for steady-state inputs. Note that this plot is on a log scale for the y-axis.

4-8. What insights are gained from looking at solutions to the branched case? Let's consider the steady-state solutions first.

Note the attenuation from point I to the branch point and the lack of attenuation from the branch point back distally to point B. Why does this happen? The current flowing into the branch point has a choice of where to go next. There is a sealed end at B which is not far from the branch point and this sealed end boundary condition forces most of the current to flow towards the soma (which looks like a 4-lane highway compared to a 1-lane dirt road to point B). What little

current flows towards B is dammed up at B, so the voltage decays little from that at the branch point.

Note the attenuation to the next branch points and terminals. Again there is more attenuation towards the soma than back out to the distal branch. Again the boundary condition of the distal branch makes the attenuation much less there.

For input at I the voltage at the soma is $< 5\%$ of that at I. What are the implications of this?

For distal inputs, there are large voltage changes locally, but small changes at the soma.

The inputs are relatively ineffective, weak, or so it would seem. However, if there are voltage-gated conductances in the dendrites, the large distal voltage changes could activate them.

Note that the attenuation from the soma for current injection there. Recall the analytic solution for a cylinder. $V(L) = V_0/\cosh(L)$ for voltage clamp at the soma to V_0 . Much of the voltage change at the soma is felt at the distal dendrites.

What does this say about inhibition? It says that inhibition is most effective when placed at the soma because the effect is felt throughout the cell.

How about excitation? Soma input will not produce the large changes in voltage seen with more distal input, so excitation at the soma is less likely to activate voltage gated conductances, suggesting that maybe excitation is better placed in the dendrites.

What about the figure for transient inputs? Note the peaks and time of peaks. The voltage change gets broader towards the soma for input to the distal terminal. Attenuation is much steeper for transient inputs. NEURON uses a `d_lambda` to set compartment size to maintain accuracy for transient inputs.